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ABSTRACT

The existence and uniqueness of viscosity solutions of possible degenerate elliptic equations in R^N is considered. For example, the equations treated include ones of the form

$$u + H(Du) - \varepsilon\Delta u = f(x) \text{ in } R^N$$

where $\varepsilon > 0$ and Du is the gradient of u, as well as fully nonlinear generalizations of this equation. Results are obtained which relate growth and continuity properties of the nonlinearity $H(p)$ and the forcing term $f(x)$ and (sometimes sharp) uniqueness classes for solutions. Existence is proved in the uniqueness classes.

AMS (MOS) Subject Classifications: 35J70, 35J15, 35F20, 35D05

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EXISTENCE AND UNIQUENESS VISCOSITY SOLUTIONS OF DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS IN R^N

Michael G. Crandall*, Richard Newcomb* and Yoshihito Tomita**

Introduction

In this paper we prove some uniqueness and some existence theorems for solutions of quasilinear (possibly degenerate) elliptic equations of the form

(E) $$u + H(x,Du) - Pu = 0 \quad \text{in} \quad R^N,$$

where $Du = (u_{x_1},\cdots,u_{x_N})$ and the linear differential operator $P = \sum_{i,j=1}^{N} p_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ has continuous coefficients $p_{i,j}$ which satisfy

(P) $$\big(p_{i,j}(x)\big) = \big(p_{j,i}(x)\big) \text{ and } 0 \le \sum_{i,j=1}^{N} p_{i,j}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for } x, \xi \in R^N.$$

for some constant Λ. The function $H: R^N \times R^N \to R$ is assumed to be continuous throughout this paper. Roughly speaking, we are interested in the interaction between some structure properties of H and the questions of existence and uniqueness. We will consider three different restrictions on H: Either H is Lipschitz continuous in p, i.e. there is a constant L such that

(H1) $$|H(x,p) - H(x,q)| \le L|p - q| \quad \text{for } x,p,q \in R^N,$$

or H is uniformly continuous, i.e. there is a modulus of continuity m such that

(H2) $$|H(x,p) - H(x,q)| \le m(|p - q|) \quad \text{for } x,p,q \in R^N,$$

or H behaves like a power of $|p|$, i.e. there is a constant K and an $m > 1$ such that

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(H3) $|H(x,p) - H(x,q)| \le K(|p|^{m-1} + |q|^{m-1} + 1)|p - q|$ for $x,p,q \in R^N$.

Suppose u is a subsolution of (E) (that is, $u + H(x,Du) - Pu \le 0$ in R^N) while v is a supersolution. In each of the three cases above we will give essentially optimal conditions on the growth of $(u - v)$ which will guarantee that the comparison result $u \le v$ holds in R^N (and hence $u = v$ if both are solutions). We will need to assume some technical restrictions on the x dependence of the coefficients of P and H(x,p). These are formulated precisely in Section 1 and vary from case to case. At the moment we will merely refer to these conditons as (TC) and note that (TC) holds if P is independent of x and H is separated, i.e. $H(x,p) = H_0(p) - f(x)$ is the difference of a function of p and a function of x.

We will prove that if (TC) holds then:

(i) If (H1) holds,

(1) $$a = \frac{-L + (L^2 + 4\Lambda)^{1/2}}{2\Lambda}, \quad b = \frac{\Lambda(N-1)a}{L + 2a\Lambda},$$

and

(2) $$\limsup_{|x| \to \infty} (u(x) - v(x))^+ e^{-a|x|}|x|^b = 0 ,$$

then

(3) $$u \le v \text{ in } R^N.$$

(ii) If (H2) holds and

(4) $$\limsup_{|x| \to \infty} (u(x) - v(x))^+ e^{-c|x|} = 0 \text{ for every } c > 0,$$

then (3) holds.

(iii) If (H3) holds, v is locally Lipschitz continuous and

(5) $$\lim_{R \to \infty} \operatorname{essential\,sup}_{|x| > R} (|Dv(x)| + ((u(x) - v(x))^+|x|^{-1}))|x|^{\frac{-1}{m-1}} = 0$$

then (3) holds. The same result holds if u is locally Lipschitz continuous and Dv is replaced by Du above.

The reader will note that we have not yet stated the notion of solution assumed above and that the regularity "v is locally Lipschitz continuous" is assumed as an extra condition in (iii) above. That is because we will be dealing with "viscosity solutions" in what follows and these solutions may be merely continuous functions. The notions of viscosity subsolutions, supersolutions and solutions are appropriate notions for second order equations $F(x,u,Du,D^2u) = 0$ where the equation is given by a function

$$F: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R},$$

where S^N is the set of symmetric $N\times N$ real matrices and F is nondecreasing in $r \in \mathbb{R}$ and nonincreasing in $A \in S^N$ when S^N carries its natural order. We recall these viscosity notions in Section 1. To emphasize this point, the reader may observe that if $P = 0$ and $H(x,p) \equiv H(x,0)$ is independent of p above, then all assumptions (including (H1), (H2) and (H3)) are satisfied provided $H(x,0)$ is continuous and, moreover, $u = -H(x,0)$ is a solution of (E) which may be nowhere differentiable. In fact, our comparison results hold for suitable classes of fully nonlinear equations $F = 0$ as is explained at the end of Section 1. The question of existence is naturally more delicate and we obtain model results for each case in Section 2. In this regard, let us mention that the proof of Theorem 5 concerning the case (H3) is, perhaps, unusual. If the nonlinearity $H(x,p)$ is a pure power $|p|^m$ and, for example, P is the Laplacian, there is a great deal of information about the corresponding equation to be found in Lions [15] and Lasry and Lions [14]. However, we do not assume any coercivity of the nonlinearity here.

We complete this introduction with some remarks. Uniqueness results for viscosity solutions for first-order partial differential equations were first obtained in M. G. Crandall and P.-L. Lions [6]. See also Crandall, Evans and Lions [5]. Soon after this the first results for the natural extension of this notion to second order equations were obained in P.-L. Lions [16] where, among other things, it is shown that the value function for some stochastic control problems is the unique viscosity solution of the associated Hamilton-Jacobi-Bellman equation.

More recently several papers treating viscosity solutions for second-order partial differential equations by analytic methods appeared. We mention the work S. Aizawa and Y. Tomita [2] in which some quasilinear elliptic equations are treated, the fundamental advance to fully nonlinear nonconvex elliptic equations made by R. Jensen [11], the subsequent important work by H. Ishii [9] as well as the improvement in method by R. Jensen, P.-L. Lions and P. E. Souganidis [13]. Most recently, the general situation has been illuminated by H. Ishii and P.-L. Lions in [10] as well as Jensen [12]. The uniqueness problem for unbounded viscosity solutions was treated in the space of continuous functions on R^N which grow at most linearly as $|x| \to \infty$ in [2] and [9]. Our uniqueness and existence results for (E) are in classes of continuous functions on R^N which grow superlinearly as $|x| \to \infty$, extending some results previously obtained for first-order equations in Crandall and Lions [7] and Ishii [8]. By relying on the results of [10] mentioned above, our uniqueness results may be proved by simple nearly classical arguments and our proofs parallel arguments in [7].

Section 1. Preliminaries and Comparison Theorems.

We begin with some remarks on the notion of viscosity solutions of second order equations in the form which we will use it. Since viscosity solutions are the principal notion of solution we will use, we will refer to them merely as solutions. Let Ω be an open subset of $\mathbf{R}^N$ and $u:\Omega \to \mathbf{R}$. We will need the following:

Definition 1: For $x \in \Omega$,

$$D^{2,+}u(x) = \{(p,A) \in \mathbf{R}^N \times S^N : \limsup_{y\to x} \frac{u(y)-u(x) - (p,y-x) - \frac{1}{2}(A(y-x),y-x)}{|y - x|^2} \le 0 \}$$

and $D^{2,-}u(x) = -(D^{2,+}(-u))(x)$.

In the above definition $(\cdot,\cdot)$ is the Euclidean inner-product on $\mathbf{R}^N$. Let $F: \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \times S^N \to \mathbf{R}$ be elliptic, i.e.

$$F(x,r,p,A) \text{ is nonincreasing in } A,$$

and nondecreasing in r. One can now define:

Definition 2: Let $u:\Omega \to \mathbb{R}$: Then u is a subsolution (respectively, supersolution) of $F = 0$ in Ω if u is upper-semicontinuous (respectively, lower-semicontinuous) and $F(x,u(x),p,A) \le 0$ (respectively, $F(x,u(x),p,A) \ge 0$) for all $x \in \Omega$ and $(p,A) \in D^{2,+}u(x)$ (respectively, $(p,A) \in D^{2,-}u(x)$). Finally, u is a solution of $F = 0$ in Ω if it is both a subsolution and a supersolution in Ω.

We remark that these notions coincide with those used in [10], [11] as applied to continuous functions. However it is common and sometimes useful to define a function which is not upper-semicontinous to be a subsolution if its upper-semicontinous envelope is a subsolution, etc.

We will use the notation

(1.1) $$B_R = \{x \in \mathbb{R}^N: |x| < R\}.$$

The technical conditions (TC) involve the following conditions:

(TC1) The nonnegative square root $(s_{i,j}(x))$ of the matrix $(p_{i,j}(x))$ is Holder continuous with exponent θ on each compact set.

(TC2) For each $R > 0$ there is a modulus of continuity ω_R such that

$$|H(x,p) - H(y,p)| \le \omega_R(|x - y|(1 + |p|))$$

for $(x,y,p) \in B_R \times B_R \times \mathbb{R}^N$.

See the end of this section for remarks on variants of these conditions. We begin by formulating results in the various cases. For the case (H1) in which H is Lipschitz continuous in p we have:

Theorem 1: Let (H1) and (TC1) hold. Let u be a subsolution of (E), v be a supersolution of (E) and (1) and (2) hold.

(i) If $\theta = 1$ and (TC2) holds, then (3) holds.

(ii) If one of u or v is Lipschitz continuous on bounded sets and $1/2 < \theta$, then (3) holds.

For the case (H2) in which H is uniformly continuous we have:

Theorem 2: Let (H2) and (TC1) hold. Let u be a subsolution of (E), v be a supersolution of (E) and (4) hold.

(i) If $\theta = 1$ and (TC2) holds, then (3) holds.

(ii) If one of u or v is Lipschitz continuous on bounded sets and $1/2 < \theta$, then (3) holds.

Finally, for the power-like case (H3) we have:

Theorem 3: Let (H3) and (TC1) hold with $\theta > 1/2$. Let u be a subsolution of (E), v be a locally Lipschitz continuous supersolution of (E) and (5) hold. Then (3) holds. Moreover, the same result is true in u is locally Lipschitz continuous and (5) holds with u in place of v.

The basic strategy in the proofs of all three results is the same. In its simplest form it runs as follows: For each $R > 0$ we construct a function $v + z_R$ on B_R with the properties

(1.2) $v + z_R$ is a supersolution of (E) on $\mathbb{R}^N$ and $u(x) \le v(x) + z_R(x)$ for $x \in \partial B_R$

and

$$\lim_{R \to \infty} z_R(x) = 0 \text{ for } x \in \mathbb{R}^N. \tag{1.3}$$

By comparison on bounded sets (see the remarks on extensions and variants in [10]) we conclude from (1.2) that

$$u(x) \le v(x) + z_R(x) \text{ for } x \in B_R \tag{1.4}$$

and then from (1.3) and (1.4) we conclude that (3) holds. The proofs of Theorems 1 and 3 run exactly as described, while Theorem 2 uses a slight adaptation.

Proof of Theorem 1. Let $w \in C^2(\mathbb{R}^N)$. We first determine sufficient conditions for $v + w$ to be a supersolution of (E). Let

$$F(x,r,p,A) = r + H(x,p) - \sum_{i,j=1}^{N} p_{i,j}(x)a_{i,j} \text{ where } A = (a_{i,j}). \tag{1.5}$$

Since w is C^2, we have

$$(1.6) \qquad D^{2,-}(v + w)(x) = D^{2,-}v(x) + (Dw(x), D^2w(x))$$

where

$$D^2w(x) = (w_{x_i, x_j}(x)).$$

Suppose $(p,A) \in D^{2,-}(v + w)(x)$; then by (1.6) $p = p_0 + Dw(x)$ and $A = A_0 + D^2w(x)$ where $(p_0, A_0) \in D^{2,-}v(x)$. Suppose now that w is convex so that $D^2w(x) \geq 0$. We now adopt the notational convention that $P(x)$ stands for the matrix $(p_{i,j}(x))$. The condition (P) states that $P(x)$ and $\Lambda I - P(x)$ are nonnegative. Hence if w is convex (so that $D^2w(x)$ is nonnegative) we have $\mathrm{trace}((\Lambda I - P(x))D^2w(x)) \geq 0$, which amounts to

$$(1.7) \qquad \Lambda \Delta w(x) \geq (Pw)(x)$$

where Pw is the differential operator P applied to w. In view of the form (1.5) of F, (H1) and (1.7) we then have

$$(1.8) \qquad F(x, v(x) + w(x), p, A) \geq F(x, v(x), p_0, A_0) + w(x) - L|Dw(x)| - \Lambda \Delta w(x).$$

Since v is a supersolution we conclude that $v + w$ is also a supersolution if

$$(1.9) \qquad w \text{ is convex and } w(x) - L|Dw(x)| - \Lambda \Delta w(x) \geq 0 \text{ in } \mathbb{R}^N.$$

We seek the radial solution of (1.9) of fastest growth. If $w(x) = G(|x|)$ where G is a nondecreasing function (1.9) becomes

$$(1.10) \qquad G(r) - \left(L + \frac{\Lambda(N - 1)}{r}\right)G'(r) - \Lambda G''(r) \geq 0 \ .$$

It is easy to see that if we insert a formal expansion

$$(1.11) \qquad G(r) = \frac{e^{ar}}{r^b} \left(1 + \sum_{j=1}^{\infty} c_j \frac{1}{r^j} \right) ,$$

in the left-hand side of (1.10) with a and b given by (1), then the coefficients c_j are uniquely recursively determined by the requirement that the resulting formal expansion is identically zero. The theory of irregular singular points (e.g., [3]) then asserts that there is a genuine solution of the equality corresponding to (1.10) which has the asymptotic expansion (1.11) near $r = \infty$. (See Caffarelli and Littman [3] and the references therein concerning the case $L = 0$). Alternatively, it is possible to show that

(1.12)
$$G(r) = \frac{e^{ar}}{r^b}\left(1 + \frac{1}{\log(r)} \right)$$

solves (1.10) in a neighborhood of ∞. These positive convex solutions of (1.10) near ∞ may be extended to a function G on $[0,\infty)$ with $G'(0) = 0$ in any convex way and then $G(r) + C$ will satisfy (1.10) everywhere as soon as C is sufficiently large. We conclude that there is a positive convex radial solution $w(x) = G(|x|)$ of (1.9) which satisfies

(1.13)
$$\lim_{|x|\to\infty} w(x)e^{-a|x|}|x|^b = 1.$$

To complete the proof of Theorem 1, set

(1.14)
$$\alpha(R) = \sup_{|x|=R} (u(x) - v(x))^+$$

and

(1.15)
$$z_R(x) = \frac{\alpha(R)}{G(R)} G(|x|) .$$

From the above considerations, $v + z_R$ and z_R have the properties (1.2) and (1.3) (since $\alpha(R)/G(R) \to 0$ as $R \to \infty$ by (2)) and the proof is complete.

Remark on the sharpness of Theorem 1: We noted in the course of proof that there is a radial solution of the equation corresponding to (1.9) (and not just the inequality) with an expansion of the form (1.11). As above, we may extend this to a subsolution of the equation corresponding to (1.9) in all of $\mathbb{R}^N$ and this subsolution will not lie below the identically zero solution. Hence the growth condition is sharp in this sense.

Proof of Theorem 2. The hypothesis (H2) implies that for every $\varepsilon > 0$ there is a constant L_ε such that

(1.16)
$$|H(x,p) - H(x,q)| \le L_\varepsilon|p - q| + \varepsilon \text{ for } x,p,q \in \mathbb{R}^N.$$

The analysis above applied in this situation shows that $v + w$ is a supersolution of (E) on $\mathbb{R}^N$ if

$$w \text{ is convex and } w(x) - L_\varepsilon|Dw(x)| - \Lambda\Delta w(x) \ge \varepsilon \text{ in } \mathbb{R}^N$$

so if we put

$$z_R(x) = \frac{\alpha(R)}{G(R)} G(|x|) + \varepsilon$$

where G is as constructed above for $L = L_\varepsilon$ we will have (1.4). Letting $R \to \infty$ and invoking (4) yield $u \le v + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we are done.

Proof of Theorem 3: Again we seek a supersolution of (E) of the form v + w. We note first that (5) implies that for each $\varepsilon > 0$ there is C_ε such that

(1.17) $$|Dv| \le \varepsilon |x|^{m^*-1} + C_\varepsilon \text{ a. e. on } R^N$$

where

(1.18) $$m^* = m/(m-1).$$

Since v is locally Lipschitz, if $(p_0,A) \in D^{2,-}v(x)$, (1.17) implies

$$|p_0| \le \varepsilon |x|^{m^*-1} + C_\varepsilon.$$

Since $(m-1)(m^*-1) = 1$ and $\varepsilon > 0$ is arbitrary, we conclude that we can choose numbers ν, $C > 0$ such that

(1.19) $$K|p_0|^{m-1} \le \nu |x| + C - K$$

and

(1.20) $$\nu m^* < 1.$$

Using (1.19) one then sees that if F is given by (1.5), (H3) holds, w is convex and $(p,A) \in D^{2,-}(v + w)(x)$, then

$$F(x, v(x) + w(x), p, A) \ge w(x) - (\nu|x| + K|Dw(x)|^{m-1} + C)|Dw| - \Lambda\Delta w$$

so we want

(1.21) $$w(x) - (\nu|x| + K|Dw(x)|^{m-1} + C)|Dw| - \Lambda\Delta w \ge 0 \text{ in } R^N.$$

We try for a solution of (1.21) near ∞ of the form $w(x) = c|x|^{m^*}$. This amounts to asking that

(1.22) $$cr^{m^*}\left(1 - \nu m^* - K(m^*)^m c^{m-1} - \frac{Cm^*}{r} - \frac{\Lambda((N-1)m^* + m^*(m^*-1))}{r^2}\right) \ge 0$$

hold for large r. In view of (1.20), (1.22) will hold for large r when we choose $c > 0$ so that $c^{m-1}K(m^*)^m < 1 - \nu m^*$. Then w may be extended from the region where (1.22) holds in a smooth convex way and increased by a constant so that $w(x) = G(|x|)$ is radial, (1.21) holds everywhere and

$$\lim_{R\to\infty} \frac{G(R)}{R^{m^*}} = c > 0. \tag{1.23}$$

Next we note that γw will also solve (1.21) if w does and $0 < \gamma < 1$. It now follows from (5) that if z_R is given by (1.15) and R is large, then $v + z_R$ is a supersolution, we have (1.3) and (1.4), and the proof is complete.

Remark on the proof of Theorem 3: We did not use the full assumption (5) in this proof. We required only the bound (1.19) when $(p_0,A_0) \in D^{2,-}v(x)$ subject to (1.20) and that

$$\lim_{|x|\to\infty} (u(x) - v(x))^+ |x|^{-m^*} = 0 . \tag{1.23}$$

Remark on the sharpness of Theorem 3: As the reader had observed, neither the statement or the proof of Theorem 3 had much to do with the nature of the second order terms in (E). For first order equations the sharpness of this result is remarked in [7] and [8]. Note also that the results in these works requires estimates on the local Lipschitz behaviour of both both u and v. The presentation here corresponds to an improvement of an argument of Newcomb (unpublished) allowing one of the functions u and v to be irregular. It may be thought that the presence of the second order terms should improve the situation, but this is not so as the next example shows. Let

$$H(p) = \begin{cases} -\left(\frac{p}{m^*}\right)^m + m^*(m^* - 1)\left(\frac{p}{m^*}\right)^{(m^*-2)/(m^*-1)} + \dfrac{(N-1)|p|}{\left(\frac{p}{m^*}\right)^{(m-1)} - \frac{m^*-2}{m^*-1}} & \text{if } |p| > m^* \\[2ex] -\dfrac{|p|^2}{2m^*(m^*-1)} + Nm^*(m^*-1) - \dfrac{m^*-2}{2(m^*-1)} & \text{if } |p| < m^* , . \end{cases}$$

Then

$$u_1(x) = \begin{cases} \left(|x| + \frac{m^* - 2}{m^* - 1}\right)^{m^*} & \text{if } |x| > \frac{1}{m^* - 1} \\ \frac{m^*(m^* - 1)}{2}|x|^2 + \frac{m^* - 2}{2(m^* - 1)} & \text{if } |x| < \frac{1}{m^* - 1} \end{cases}$$

and $u_2 = -Nm^*(m^* - 1) + (m^* - 2)/2(m^* - 1)$ are both solutions of

$$u + H(Du) - \Delta u = 0$$

in R^N with the growth $|x|^{m^*}$.

We end this section with several other remarks.

Remark on dependence on forcing terms: Suppose instead of (E) u is a solution of $u + H(x,Du) - Pu < f(x)$ and v is a solution of $v + H(x,Dv) - Pv > g(x)$ where f and g are continuous and one of the above results applies to (E) under growth assumptions satisfied by $u - v$. Then $u - \sup_x(f(x) - g(x))$ and v are sub and supersolutions of the same problem and we conclude that

$$u - v < \sup_x (f(x) - g(x)) . \tag{1.24}$$

Remarks on the fully nonlinear case: We want to consider a general equation $F = 0$ and keep in mind the special case $F(x,r,p,A) = r + H(x,p) - \sum_{i,j=1}^{N} p_{i,j}(x)a_{i,j}$ where $A = (a_{i,j})$. The conditions on F which correspond to (P), (H1) - (H3) are (F1), (F2), (F3) below. F is elliptic and for $x,p,q \in R^N$, $r \in R$, $s > 0$, $A \in S^N$ and $B \in S^{N+}$ (the nonegative symmetric matrices) we have:

(F1) $\quad F(x,r + s, p + q, A + B) > F(x,r,p,A) + s - L|q| - \Lambda\text{trace}(B).$

(F2) $\quad F(x,r + s, p + q, A + B) > F(x,r,p,A) + s - m(|q|) - \Lambda\text{trace}(B),$

and

(F3) $\quad F(x,r+s,p+q,A+B) > F(x,r,p,A) + s - K(|q|^{m-1}+|p|^{m-1}+1)|q| - \Lambda\text{trace}(B).$

It is clear that analogous uniqueness results to Theorems 1 - 3 hold for fully nonlinear equations with (F1), (F2), (F3) in place of (H1), (H2), (H3) if one has comparison results for the equations on balls.

Remark on the strictly elliptic case: We did not use any assumptions of nondegeneracy of the second order operator P. If indeed the matrix P(x) is nonsingular for every x, then the equation is strictly elliptic on bounded sets in the sense defined in Ishii - Lions [10] and the technical conditions can be relaxed. The reader will be able to formulate the required conditions upon perusing [10].

Section 2: Existence

The simplest method to obtain existence for (E) is Perron's. As applied here, it works as follows: Suppose we can find a continuous supersolution $\bar{u}$ and a continuous subsolution $\underline{u}$ such that $\bar{u} \geq \underline{u}$ and $(\bar{u} - \underline{u})^+$ is bounded in such a way that a comparison result (which varies from case to case) applies. Then it is known that

$$u(x) = \sup\{v(x) : v \text{ is a subsolution of (E) and } \underline{u} \leq v \leq \bar{u} \text{ on } R^n\}$$

is (lower-semicontinuous and) a subsolution whose lower-semicontinuous envelope u_* is a supersolution. Since $u - u_* \leq \bar{u} - \underline{u}$ we conclude that $u \leq u_*$, so $u = u^*$ and u is a solution. For example, if (H2) holds, it is enough to produce $\bar{u}$, $\underline{u}$ which such that $(\bar{u} - \underline{u})^+$ has slower than exponential growth (i.e., (4) holds with $u = \bar{u}$, $v = \underline{u}$). An easy case in which we can do this is the following:

Proposition 4: Let (H2), (TC1) and (TC2) hold with $\theta = 1$. Suppose that there are constants $C, \mu > 0$ such that

$$|H(x,0)| \leq C(|x|^{\mu} + 1). \tag{2.1}$$

Then (E) has a solution u such that $u/(1 + |x|^{\mu})$ is bounded.

Proof. Let L be such that

$$|H(x,p) - H(x,0)| \leq L|p| + 1 \text{ for } x,p \in R^N. \tag{2.2}$$

Then using (2.1) and (2.2) one sees immediately that a convex function $\bar{u}(x)$ will be a super-solution if

$$\bar{u} - L|D\bar{u}| - \Lambda\Delta\bar{u} \geq C(|x|^{\mu} + 1) + 1 \text{ on } R^N. \tag{2.3}$$

Moreover, $\bar{u}(x) = (C + 1)|x|^{\mu}$ satisfies (2.3) for large $|x|$. We extend this solution from the neighborhood of ∞ in which (2.3) holds in any smooth, radial and convex way to all R^N and then increase it by a constant so that (2.3) holds everywhere. A (nonpositive) subsolution can be constructed in the analogous way and the proof is complete.

Since (H1) implies (H2), the above result applies when (H1) holds as well. However, we can do better. If (2.1) is relaxed to

$$(2.4) \qquad |H(x,0)| < C\frac{e^{a|x|}}{|x|^{b+1+\delta}} \text{ for } |x| > 1$$

where a,b are given by (1) and $\delta > 0$ and still ask that (2.2) hold, then we can again find suitable sub and supersolutions. Indeed, if $\bar{u} = G(r)$ is radial and convex, and (2.2) and (2.4) hold, then it $\bar{u}$ is a supersolution if

$$(2.5) \qquad G(r) - \left(L + \frac{N-1}{r}\right)G'(r) - G''(r) - C\frac{e^{ar}}{r^{b+1+\delta}} - 1 > 0.$$

As usual, it is enough to solve in a neighborhood of ∞. We try $G(r) = e^{ar}r^{-(b+\delta/2)}$, in which case the left hand side of (2.5) becomes

$$\frac{e^{ar}}{r^{b+(\delta/2)}}\left((L + 2a\Lambda)\frac{\delta}{2} - \frac{\Lambda(\mu)(\mu+1)}{r} - \frac{C}{r^{(\delta/2)}} - e^{-ar}r^{b+(\delta/2)}\right),$$

so (2.5) indeed holds in a neighborhood of ∞. A (nonpositive) subsolution of similar form is constructed in the same way.

We turn to the question of existence under (H3). The situation is now quite different, for the corresponding comparison result Theorem 3 requires that one of the functions u or v admit a suitable estimate on its derivative and we cannot use Perron's method as above. The proof below is interesting in that the unique solution produced is not shown to be differentiable at all; indeed, we leave open the question of regularity. We will give a model result for this case when (E) is simplified to the problem

(E)' $$u + H(Du) - Pu = f(x)$$

which is free of x dependence except in the function f - in particular, we assume that P has constant coefficients.

<u>Theorem 5:</u> Let $H(p)$ satisfy (H3), P have constant coefficients, $1 < \mu < m^*$ and there be a constant c_0 such that

(2.6) $$|f(x) - f(y)| < c_0 R^{(\mu - 1)}|x - y| \text{ for } x,y \in B_R,\ R > 1.$$

Then (E)' has a unique solution u such that $u(x)/(1 + |x|^\mu)$ is bounded.

<u>Proof:</u> Let

$$T_n x = \begin{cases} x & \text{if } |x| < n, \\ \dfrac{nx}{|x|} & \text{if } |x| > n, \end{cases}$$

be the radial projection of R^N on B_n and approximate (E)' by the problem

(2.7) $$U + H(T_m DU) - PU = f(T_n x)$$

where $m, n > 0$. (2.7) has a unique bounded solution U (in view of the constant sub and supersolutions) by the discussion of the case (H1) above (as well as many other ways). Moreover, for $y \in R^N$, $U(\cdot + y)$ solves the same problem with $f(T_n(\cdot + y))$ in place of $f(T_n \cdot)$, so by Remark 2 of Section 1 and (2.6)

(2.8) $$|U(x + y) - U(x)| < \sup_x |f(T_n(x + y)) - f(T_n x)| < c_0 n^{\mu-1}|y|$$

It follows that the first order sub and super derivatives of U are bounded by $c_0 n^{\mu-1}$, so if m is large $u_n = U$ also solves

$$u_n + H(Du_n) - Pu_n = f(T_n x).$$

We have begun indexing by n as we now want to show the u_n converge to a limit u with the desired properties. Since it is easy to produce sub and supersolutions of (E)' which are multiples of $(1 + |x|^2)^{\mu/2}$ there is a constant c such that

(2.9) $$|u_n(x)| < (c/2)|x|^\mu \text{ for } |x| > 1.$$

From (2.8) (for u_n) we can also assume that

$$(2.10)\qquad \text{if } (p,A) \in D^{2,-}u_n(x) \text{ then } |p| < c_0 n^{\mu-1}$$

and using this in the usual way we find that if z is smooth then $u_n + z$ is a supersolution of (E)' on B_n if z is convex and

$$(2.11)\qquad z - (K|Dz|^{m-1} + K(c_0 n^{\mu-1})^{m-1} + K)|Dz| - \Lambda\Delta z \geq 0 \text{ on } B_n.$$

Now we choose γ so that

$$(2.12)\qquad \gamma > \frac{m^* \mu}{m^* - \mu}$$

and consider the function

$$(2.13)\qquad z_n(x) = cn^{\mu-\gamma}(n^{(2\mu/m^*)} + |x|^2)^{\gamma/2}.$$

Clearly z_n is convex and smooth. Moreover, using (2.9), we see that

$$(2.14)\qquad z_n(x) \geq |u_n(x) - u_k(x)| \text{ for } x \in \partial B_n \text{ and all } k, n.$$

We also claim that z_n satisfies (2.11) if n is large. The existence assertion will follow from this, since once this last claim is verified comparison on B_n implies

$$u_n(x) - u_k(x) \leq z_n(x) \text{ in } B_n \text{ for } n \leq k$$

because the equations for u_n and u_k agree on B_n if $n \leq k$. Moreover, the analogous estimate from below is proved in the same way, so in fact

$$(2.15)\qquad |u_n(x) - u_k(x)| \leq z_n(x) \text{ in } B_n \text{ for } n \leq k.$$

Finally, (2.12) is equivalent to $\mu - \gamma + \mu\gamma/m^* < 0$, which means that

$$(2.16)\qquad \lim_{n\to\infty} z_n(x) = 0 \text{ uniformly for bounded } x$$

so (2.15) implies that the u_n are Cauchy on bounded sets. Therefore the u_n converge locally uniformly to the desired solution. Thus the proof of existence is complete once we show that $z = z_n$ solves (2.11). To this end we write $z_n(r)$ with the obvious meaning, and calculate:

$$(2.17)\qquad z_n'(r) = c\gamma n^{\mu-\gamma}(n^{(2\mu/m^*)} + r^2)^{(\gamma/2 - 1)}r,$$

$$(2.18)\qquad z_n''(r) = c\gamma n^{\mu-\gamma}(n^{(2\mu/m^*)} + r^2)^{(\gamma/2 - 2)}((\gamma - 1)r^2 + n^{(2\mu/m^*)}).$$

From this we see that $z_n(r)$ is convex and nondecreasing, so $z_n(x)$ is convex. Hereafter we let C denote various constants which vary from occurence to occurence in a harmless way. Since $\mu < m^*$, $2\mu/m^* < 2$ and

$$n^{(2\mu/m^*)} + r^2 < Cn^2 \text{ for } r < n.$$

Thus, using (2.17),

(2.19) $$|Dz_n| < Ccn^{\mu-1} \text{ on } B_n$$

and so

(2.20) $$K(|Dz_n|^{m-1} + (c_0{}^{m-1}n^{(\mu-1)(m-1)} + 1) < Cn^{(\mu-1)(m-1)} \text{ on } B_n.$$

As regards Δz_n, from (2.17), (2.18) we find

$$\Delta z_n = c\gamma(\gamma-2)n^{\mu-\gamma}(n^{(2\mu/m^*)} + |x|^2)^{(\gamma/2 - 2)}|x|^2$$
$$+ c\gamma n^{\mu-\gamma}N(n^{(2\mu/m^*} + r^2)^{(\gamma/2 - 1)} < cn^{\mu-\gamma}(n^{(2\mu/m^*)} + r^2)^{(\gamma/2 - 1)}$$

on B_n. Thus, setting

(2.21) $$\theta = n^{(2\mu/m^*)} + |x|^2,$$

we have

$$z_n - (K|Dz_n|^{m-1} + K(c_0{}^{\mu-1})^{m-1} + K)|Dz_n| - \Lambda\Delta z_n >$$
$$cn^{\mu-\gamma}\theta^{\gamma/2} - Cn^{(\mu-1)(m-1)}|Dz_n| - \Lambda cn^{\mu-\gamma}\theta^{(\gamma/2 - 1)}$$
$$= cn^{\mu-\gamma}\theta^{(\gamma-2)/2}(\theta - Cn^{(\mu-1)(m-1)}\theta^{1/2} - \Lambda C)$$

on B_n. Now

(2.22) $$\theta - Cn^{(\mu-1)(m-1)}\theta^{1/2} - \Lambda C > 0$$

provided $\theta^{1/2}$ is larger than the greatest root t_0 of the equation

$$t^2 - Cn^{(\mu-1)(m-1)}t - \Lambda C = 0$$

which has the form

$$t_0 = \frac{Cn^{(\mu-1)(m-1)} + (Cn^{(\mu-1)(m-1)2} + 4\Lambda C)^{1/2}}{2} < Cn^{(\mu-1)(m-1)}$$

However, from (2.21) we see that $\theta^{1/2} > n^{(\mu/m^*)}$ and from $\mu < m^*$ it follows that

$$(\mu - 1)(m - 1) < \mu/m^*$$

so (2.22) holds on B_n if n is sufficiently large. The proof of existence is complete.

It remains to discuss the uniqueness. The uniqueness is in fact a consequence of the above arguments, since if u is a solution of (E)' with the asserted growth, it solves the same equation as u does on B_n and the estimates made above on $u_n - u_k$ apply equally well to $u_n - u$. Thus u is the limit of the u_n, and the uniqueness follows.

Remarks on Theorem 5: The result may be extended to an equation $u + H(x,Du) - Pu = f(x)$ with x dependence in H with a similar proof if the local Lipschitz behaviour of H in x is like that imposed on f(x) independently of p. We also note that the corresponding existence result in [7] imposed a coercivity condition on H(p) from which the local Lipschtiz behaviour was deduced. The proof in [7] and ours are totally distinct.

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